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ON THE BUCKLING OF SYMMETRIC STRUCTURAL SYSTEMS WITH FIRST AND SECOND ORDER IMPERFECTIONS

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Abstract—The stability of symmetric structural systems with first order and second order imperfections is investigated. The analysis concerns a discrete system with n degrees of freedom. Resulting expressions relating reduction in critical load to the magnitudes of the imperfections indicate that a complex interaction exists between these quantities. Second order imperfections have a relatively minor effect on the critical load. Experiments on a rigidly jointed roof truss are discussed. The experimental results show good qualitative agreement with theory.

1. INTRODUCTION

IN the past few decades much progress has been made in the study of the stability of imperfect structural systems in the initial post-buckling range. The work by Koiter [1], who provided a general nonlinear theory for stability in the context of continuum elasticity, must be regarded as the basis for the study of any problem of this type. In more recent times Roorda [2] and Thompson [3] have presented more detailed studies of the bifurcation modes, originally predicted by Koiter, that arise in elastic stability theory, namely the asymmetric mode, the stable-symmetric mode and the unstable-symmetric mode. In these works (which are concerned with discrete structural systems with n degrees of freedom) as well as in Koiter's work only first order imperfections were considered, these being the most influential ones that can occur. It was shown that the relationship between this type of imperfection and the buckling load of the structure follows indeed Koiter's general one-half and two-thirds power law for asymmetric and symmetric systems respectively. It is also worthwhile to note that the experimental results on a continuous asymmetric system reported in Ref. [2] are not only qualitatively, but also quantitatively in full agreement with the general theory [4].

In a later work by Roorda [5] second order imperfections, as well as the first order type, were introduced. The complex interaction of these different classes of imperfections and their combined effect on the buckling load were studied with specific reference to asymmetric systems. Second order imperfections were found to be of relatively minor importance in this case.

The purpose of the present work is to study the effect of second order imperfections on the buckling of symmetric systems. The theory is developed for a system with n degrees of freedom, with emphasis on unstable-symmetric systems. Certain experimental results obtained from a simple roof truss are included to substantiate the theoretical conclusions.

2. EQUILIBRIUM

In the case of a discrete, conservative, imperfect structural system there exists a potential energy function V which depends on the load parameter, P, the generalized co-ordinates,

$Q_i (i = 1, 2, 3, \ldots, n)$ and the imperfection parameters ε and β, i.e.

$$V = V(P, Q_i, \varepsilon, \beta). \tag{1}$$

The potential energy V is a single valued function of these variables. The potential energy of the idealized (perfect) structural system can be obtained by setting $\varepsilon = \beta = 0$ in equation (1) i.e.

$$V = V(P, Q_i, 0, 0). \tag{2}$$

In the region of configuration space, R, that is of interest the function V and its partial derivatives with respect to P, Q_i, ε and β, to any required order, are postulated to exist and to be continuous throughout R.

The necessary and sufficient condition that the system be in a state of equilibrium is that V must be stationary and hence $V_i = 0$ $(i = 1, 2, 3, \ldots, n)$ at an equilibrium configuration Q_{ie} in R. The subscript on V here and in the sequel indicates partial differentiation with respect to Q_i. For the idealized structural system we have n equilibrium equations $V(P, Q_i, 0, 0)_i = 0$ $(i = 1, 2, 3, \ldots, n)$, which may be solved to yield a number of solutions of the type $Q_i = Q_i(P)$ each of which defines an equilibrium path for the idealized system in the $(n+1)$ dimensional $P-Q_i$ space.

Suppose one of these solutions, namely $Q_{iB} = Q_{iB}(P)$ defines an equilibrium path that passes through the region of interest R. This path will usually be the unbuckled equilibrium path and will subsequently be called the *basic state*. Q_{iB} must be single valued in the region R to exclude the possibility that P reaches a maximum or minimum in R. Without this provision the value of $Q_{iB}(P)$ would be ambiguous and subsequent analysis would break down.

We can now apply the transformation

$$Q_i = Q_{iB} + q_i \tag{3}$$

and the non-singular, orthogonal transformation of variables

$$q_i = \alpha_{ij}(P) u_j \tag{4}$$

to obtain a new potential function

$$T(P, u_j, \varepsilon, \beta) = V[P, Q_{iB}(P) + \alpha_{ij}(P) u_j, \varepsilon, \beta]. \tag{5}$$

The transformations (3) and (4) are chosen in a special way, giving the function $T(P, u_j, \varepsilon, \beta)$ the following special properties only: (i) $T_i = T_{pi} = T_{ppi} = \ldots = 0$ for all i; (ii) $T_{ij} = T_{pij} = \ldots = 0$ for all i and j, $i \neq j$, where the subscripts on T indicate partial differentiation with respect to $u_i, u_j, \ldots$, and P, and where the partial differentials are evaluated on the basic state of equilibrium. The first of these properties follows immediately from the fact that $V_i = 0$ for all values of P on the basic state. The second is of course due to the choice of coefficients in the transformation matrix $\alpha_{ij}(P)$. In essence, the second variation of T is diagonalized at all values of the load P on the basic state of equilibrium, and hence $u_j (j = 1, 2, 3, \ldots, n)$ are the principal co-ordinates.

We now proceed to find the equilibrium states for the imperfect system in the region R about a point B on the basic state, at which $P = P_0$, $u_i = 0$ $(i = 1, 2, 3, \ldots, n)$ and $\varepsilon = \beta = 0$. Applying Taylor's Theorem to the function T and using the equilibrium condition $T_i = 0$ (a direct consequence of $V_i = 0$) yields the following n equilibrium equations:

$$\frac{\partial t}{\partial u_i} = t_i = T_{ii}u_i + T_{i\varepsilon}\varepsilon + T_{i\beta}\beta$$
$$+\frac{1}{2!}[T_{ijk}u_ju_k + 2T_{pii}pu_i + T_{i\varepsilon\varepsilon}\varepsilon^2 + T_{i\beta\beta}\beta^2 + 2T_{ij\varepsilon}u_j\varepsilon$$
$$+2T_{ij\beta}u_j\beta + 2T_{pi\varepsilon}p\varepsilon + 2T_{pi\beta}p\beta + 2T_{i\varepsilon\beta}\varepsilon\beta] \tag{6}$$
$$+\frac{1}{3!}[T_{ijkl}u_ju_ku_l + \ldots] + \ldots = 0.$$

Here the increment in P is denoted by p and the subscripts p, ε and β on T indicate partial differentiation with respect to p, ε and β. Lower case t indicates the small increment in potential energy, i.e.

$$t = T(P_0 + p, u_i, \varepsilon, \beta) - T(P_0, 0, 0, 0).$$

The usual summation convention has been adopted.

In equation (6) the coefficients T_{ii} may be recognized as the stability coefficients. When one or more of the stability coefficients are zero the basic state of equilibrium is critical. Suppose we restrict our analysis to the primary critical points for which only one of the stability coefficients vanishes, all others remaining positive. Suppose in fact that $T_{11} = 0$, $T_{rr} > 0$ for $r \neq 1$. Clearly u_1 is then the critical principal co-ordinate and will be called the deflection of the system. Roorda [5] has shown that in this case the coefficient T_{111} plays an important role in the behaviour of the structural system in the neighbourhood of a critical point. In fact it was shown that T_{111} determined the initial slope of the idealized post-buckling path in an *asymmetric* system. When T_{111} vanishes, this slope is zero and it is to be expected that the initial curvature will play an important part in such systems. The initial, idealized post-buckling path will be symmetrical about the basic equilibrium path. Therefore a system with these special properties is called a *symmetric* system and is the subject of the present paper.

In the analysis so far, two types of imperfections were introduced, namely ε and β, without any real distinction being made between them. Considering the possibility that ε and β belong to different classes of imperfections we now suppose that $T_{1\varepsilon} = 0$ at the point of critical equilibrium on the basic state whereas $T_{1\beta} \neq 0$. In effect β is a first order, and ε a second order imperfection.

Clearly setting $T_{11} = T_{111} = T_{1\varepsilon} = 0$ in equation (6) results in two types of equilibrium equations for the imperfect symmetric system. The first type, a single equation, governs the behaviour of the critical principal co-ordinate u_1 and the second type, a set of $(n-1)$, governs the behaviour of the non-critical principal co-ordinates u_r, $r \neq 1$. Due to the incremental character of the variables p, u_i, ε and β the simplifying procedure of neglecting all second order terms that contain variables that also appear as first order terms, and neglecting third order terms that depend on combinations of variables that also appear as second order terms, etc, is justified. The resulting simplified equilibrium equations may be written as

$$t_1 = T_{1\beta}\beta + \frac{1}{2!}[2T_{11r}u_1u_r + T_{1rs}u_ru_s + 2T_{p11}pu_1 + T_{1\varepsilon\varepsilon}\varepsilon^2$$

$$+2T_{1j\varepsilon}u_j\varepsilon + 2T_{p1\varepsilon}p\varepsilon] + \frac{1}{3!}T_{1111}u_1^3 = 0 \quad (7a)$$

$$t_r = T_{rr}u_r + T_{r\varepsilon}\varepsilon + T_{r\beta}\beta + \tfrac{1}{2}T_{r11}u_1^2 = 0 \qquad (r \neq 1,\ s \neq 1). \quad (7b)$$

It is important to remember the provision which underlies the above simplifications, i.e. all relevant coefficients in the potential energy expansion, except T_{11}, $T_{1\varepsilon}$ and T_{111}, have non-zero values. Violation of this provision may affect the validity of these simplifications.

Solving equation (7b) for u_r and substituting in equation (7a), neglecting terms whose contributions are small compared to others, provides us with an equation in u_1 alone, namely,

$$T_{1\beta}\beta + \frac{1}{2!}[2T_{p11}pu_1 + 2Au_1\varepsilon + 2T_{p1\varepsilon}p\varepsilon + B\varepsilon^2] + \frac{1}{3!}\tau u_1^3 = 0 \quad (8)$$

in which the constants τ, A and B are functions of the T coefficients only. Equation (8) is exact in the limit as u_i, p, ε and β tend to zero and presents a good approximation for small values of these parameters.

For constant ε and β, equation (8) describes a curve on a p–u_1 plot which approaches two "asymptotes" whose intersection occurs at the point (u_{10}, p_0) where

$$u_{10} = -\frac{T_{p1\varepsilon}}{T_{p11}}\varepsilon \quad (9a)$$

$$p_0 = -\frac{1}{T_{p11}}\left(A\varepsilon + \frac{T_{p1\varepsilon}^2}{2T_{p11}^2}\tau\varepsilon^2\right). \quad (9b)$$

Shifting the origin to the point (u_{10}, p_0) by setting

$$u_1 = u_{10} + \bar{u} \quad (10a)$$

$$p = p_0 + \bar{p} \quad (10b)$$

and neglecting a small cubic term in ε, the equilibrium equation is transformed to the more convenient form

$$\tfrac{1}{6}\tau\bar{u}_1^3 + T_{p11}\bar{p}\bar{u}_1 - \tfrac{1}{2}\tau\frac{T_{p1\varepsilon}}{T_{p11}}\varepsilon\bar{u}_1^2 + T_{1\beta}\beta + C\varepsilon^2 = 0 \quad (11a)$$

where C is given by

$$C = \tfrac{1}{2}\left(B - 2\frac{T_{p1\varepsilon}}{T_{p11}}A\right). \quad (11b)$$

In terms of the new variables, the asymptotes assume the simple forms

$$\bar{u}_1 = 0 \quad (12a)$$

$$\tfrac{1}{6}\tau\bar{u}_1^2 + T_{p11}\bar{p} - \tfrac{1}{2}\tau\frac{T_{p1\varepsilon}}{T_{p11}}\bar{u}_1\varepsilon = 0. \quad (12b)$$

For the ideal structural system both ε and β are zero thus giving rise to an equilibrium equation which has the two solutions

$$u_1 = 0 \tag{13a}$$

$$\tfrac{1}{6}\tau u_1^2 + T_{p11}p = 0. \tag{13b}$$

The first is the trivial solution that corresponds to the basic state. The second solution represents another equilibrium path that intersects the basic state at the critical point B, i.e. the point of bifurcation. In a plot of p against u_1, this is a parabola with curvature proportional to τ. If τ is positive, as in a column or plate type structure, the system is *stable-symmetric*. If, on the other hand, τ is negative as in an arch type structure, then the system is *unstable-symmetric*. These two types of structures have been treated by Roorda [2] with respect to β imperfections only, and experimental results were presented.

It can be seen that except for the small linear term in $\bar{u}_1$, in equation (12), which is present by virtue of the ε imperfection, the asymptotes correspond closely with the ideal solutions. In the limit, when $\varepsilon = 0$ but $\beta \neq 0$, the equilibrium equation (11) has as asymptotes the solutions of the ideal structural system.

3. STABILITY

To determine the stability or instability of the equilibrium states of the imperfect system we must examine the second variation of the potential energy with respect to small displacements from the equilibrium states. It can be shown that equilibrium states in the neighbourhood of the primary critical point B are stable if $|t_{ij}| > 0$ and unstable if $|t_{ij}| < 0$. Evaluating this determinant, and disregarding terms that are of higher order, yields

$$|t_{ij}| = \left(t_{11} - \frac{T_{r11}^2}{T_{rr}}u_1^2\right) \cdot \prod_{k=2}^{n} T_{kk}, \qquad \text{(sum over } r\text{)}. \tag{14}$$

Thus, since T_{kk} are all non zero and positive, the stability criterion is given by

$$t_{11} - \frac{T_{r11}^2}{T_{rr}}u_1^2 \;\begin{matrix} > \\ = \\ < \end{matrix}\; 0 \qquad \begin{matrix} \text{stable} \\ \text{for critical equilibrium.} \\ \text{unstable} \end{matrix} \tag{15}$$

Proceeding with the differentiation of t, neglecting the small terms dependent on β, and expressing the result in terms of the new variables $\bar{u}_1$ and $\bar{p}$ yields the following stability criterion which is consistent with the present order of approximation:

$$\tfrac{1}{2}\tau\bar{u}_1^2 + T_{p11}\bar{p} - \tau\frac{T_{p1\varepsilon}}{T_{p11}}\bar{u}_1\varepsilon \;\begin{matrix} > \\ = \\ < \end{matrix}\; 0 \qquad \begin{matrix} \text{stable} \\ \text{for critical equilibrium.} \\ \text{unstable} \end{matrix} \tag{16}$$

The equality of (16) may be obtained directly by implicit differentiation of the equilibrium equation (11) with respect to $\bar{u}_1$ and setting $d\bar{p}/d\bar{u}_1 = 0$ as the condition for a critical point, thus indicating that extremum points on the equilibrium paths are critical states of equilibrium. This equality may be designated as the stability boundary which separates the stable zone from the unstable zone. In a $\bar{p}$–$\bar{u}_1$ plot, all equilibrium states lying above the stability boundary are unstable while those lying below are stable. All equilibrium states lying on the boundary are critical. It may also be noted here that the slope of the asymptote given by equation (12b) is equal to half the slope of the stability boundary at the point $\bar{u}_1 = \bar{p} = 0$. This result is in agreement with the findings for the asymmetric system outlined in an earlier paper [2].

Figure 1 shows the equilibrium paths in the vicinity of the primary critical point for an unstable-symmetric system. The curves are drawn for constant ε. The β imperfection is

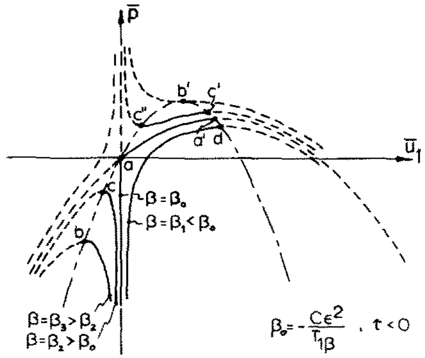

FIG. 1. Equilibrium paths.

allowed to vary, giving rise to a number of equilibrium paths. The stability boundary is indicated as a dash-dotted line while stable and unstable portions of the equilibrium paths are shown as solid and broken lines respectively. When

$$\beta = \beta_0 = -\frac{C}{T_{1\beta}}\varepsilon^2 \tag{17}$$

there are two distinct equilibrium paths that intersect at a point of bifurcation. It must be remembered that Fig. 1 is valid only for the case $\varepsilon \neq 0$. When $\varepsilon = 0$ the extremum of the asymptote would lie at the origin (i.e. $\bar{u}_1 = u_1 = 0$, $\bar{p} = p = 0$). It should also be emphasized here that for $\beta = 0$, $\varepsilon \neq 0$, there will in general not be a point of bifurcation.

4. CRITICAL STATES

Critical states of equilibrium lie at the points where the equilibrium paths cross the stability boundary. We therefore have to solve the stability equation (16) and the equilibrium equation (11) simultaneously to find their point of intersection ($\bar{u}_1^*$, $\bar{p}^*$). Due to the extreme nonlinearity of these equations, an exact solution cannot be found very easily. To gain, at least, an insight into the behaviour of the system in the limit as the imperfections assume infinitesimally small values, only the most important terms (i.e. those with lower power) are retained at each stage in the calculations. The following relationships are then obtained

$$\tfrac{1}{3}\tau\bar{u}_1^{*^3} = T_{1\beta}\beta + C\varepsilon^2 \tag{18a}$$

$$8T_{p11}^3\bar{p}^{*^3} = -9\tau(T_{1\beta}\beta + C\varepsilon^2)^2. \tag{18b}$$

It is evident that the $(\bar{u}_1^*, \bar{p}^*)$–ε–β relationships are very complex. Since a knowledge of the critical load of a structure is usually more important than a knowledge of the critical deflection, only the behaviour of the critical load will be studied here in greater detail.

With reference to equation (18b), a three dimensional diagram may be drawn illustrating the interaction of the two classes of imperfection and their combined effect on the critical load. Equation (18b) defines a surface in $\bar{p}^*$–ε–β space that has a sharp curved ridge lying in the ε–β plane. Figure 2 depicts such a surface for an unstable symmetric system for the three cases $C \gtreqless 0$. This diagram clearly illustrates the effect of changing the sign of C. It appears that for a symmetric structural system a critical point occurs for all possible combinations of ε and β. This result is in contrast to the case of the asymmetric system in which certain restrictions had to be placed on the values of ε and β to ensure buckling [5].

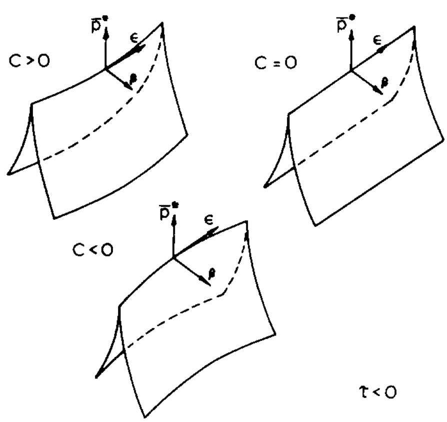

FIG. 2. Stability surfaces.

Writing equation (18b) in a more explicit form, with regard to equation (9) and equation (10), yields

$$p^* = -\frac{1}{T_{p11}}\left[A\varepsilon + 3\left(\frac{\tau}{24}\right)^{\frac{1}{3}}(T_{1\beta}\beta + C\varepsilon^2)^{\frac{2}{3}}\right]. \tag{19}$$

It is seen here that the linear dependence of p^* on ε is modified by term of order $\varepsilon^{\frac{4}{3}}$ essentially due to curvature of the ridge in the surfaces in Fig. 2. Curves corresponding to the intersection of a plane β = constant with the surface for $C < 0$ are sketched in Fig. 3(a) for various values of β. The dashed line represents the condition given by equation (17).

Clearly there is, at least initially, a linear relationship between p^* and ε so that the effect of a small second order imperfection on the critical load will be relatively unimportant. For this reason the second order imperfection may be called a minor imperfection. It is seen that for non-zero values of the first order imperfection β, there may occur two cusps in the p^*–ε relationship at which the structure buckles at a point of bifurcation.

The dependency of p^* on β follows a two-thirds power law and is depicted in Fig. 3(b) for various constant values of ε. In essence, a vanishing small first order imperfection

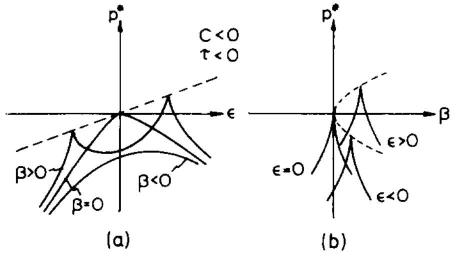

FIG. 3. (a) p^*–ε curves for constant values of β. (b) p^*–β curves for constant values of ε.

has a proportionately larger effect on the critical point, and may therefore be labelled as a major imperfection. The locus of the cusp in these curves is a parabola initially, and may be obtained by eliminating ε from equation (17) and equation (19). It is given by

$$T_{p11}^2 p^{*2} = -A^2 \frac{T_{1\beta}}{C} \beta. \tag{20}$$

As already mentioned, the slope of the p^*–ε curves is infinite in certain instances. This means that the minor imperfection, ε, takes on some of the characteristics of the major imperfection, β, in that a small change in ε will produce an appreciable change in p^*. However this effect is extremely localized and far overshadowed by small changes in β.

In view of the drastic simplifications employed in deriving equation (18) and subsequent equations, it is perhaps of benefit here to examine, in a qualitative manner, the results to be expected if higher order terms are retained. With reference to the equilibrium paths drawn in Fig. 1, the typical form of a more accurate $\bar{p}^*$–β curve for constant, non-zero ε can easily be deduced and is drawn in Fig. 4. The effect of higher order terms in the analysis

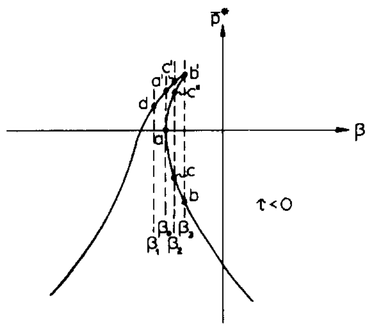

FIG. 4. $\bar{p}^*$–β curve depicting the higher order effect of ε (Derived from Fig. 1).

is therefore to "bend" the point of the cusp to the right or to the left depending on the constants in the equations. The corresponding critical points in Fig. 1 and Fig. 4 are indicated by the letters $aa'bb'cc'c''$. It is notable that in the small area around the point of bifurcation "a" a symmetric system with second order imperfections has the same characteristics as an asymmetric system in that the "ideal" post-buckling path ($\beta = \beta_0$) has a non-zero slope. The $\bar{p}^*$–β plot will therefore be parabolic as it passes through the point "a" [2].

When $\beta = \beta_2$ there are three possible critical points, two of which merge into one when $\beta = \beta_3$. For $\beta > \beta_3$ or $\beta < \beta_0$ only one critical point is possible and the curve approaches the cusp predicted by the simplified theory embodied in equation (19). The cusps appearing in the p^*–ε relationship will be modified in a similar manner if higher order terms are included.

In Fig. 4 the portion of the curve to the right of, and above, the point "a" represents critical points on the complimentary equilibrium paths and is therefore of academic interest only. The remaining part stems from the natural equilibrium paths (i.e. those obtained by gradually increasing the load on the structure from zero) and are therefore of practical significance.

5. EXPERIMENTS

The experiments were performed on a rigidly jointed roof truss with a span of 48 in. The members were made from high strength steel strip with a nominal cross-section of 1 in. × $\frac{1}{16}$ in. Two vertical loads of equal magnitude were applied through knife-edges at the two half-way points on the upper chord of the truss. A moveable knife seat device was used to allow small movements of the points of loading away from the joints. A distinction is drawn between symmetrical and anti-symmetrical eccentricities. If both load points are displaced inward or outward equal amounts, d_s, the eccentricity is symmetrical. If both load points are moved to the left or to the right equal amounts, d_a, the eccentricity is anti-symmetrical. The positive directions of d_s and d_a are outward and to the right respectively.

A schematic diagram of the frame and the loading arrangement is shown in Fig. 5. The spring balance and screw-jack loading device is semi-rigid, allowing unstable portions of the load-deflection curves to be traced.

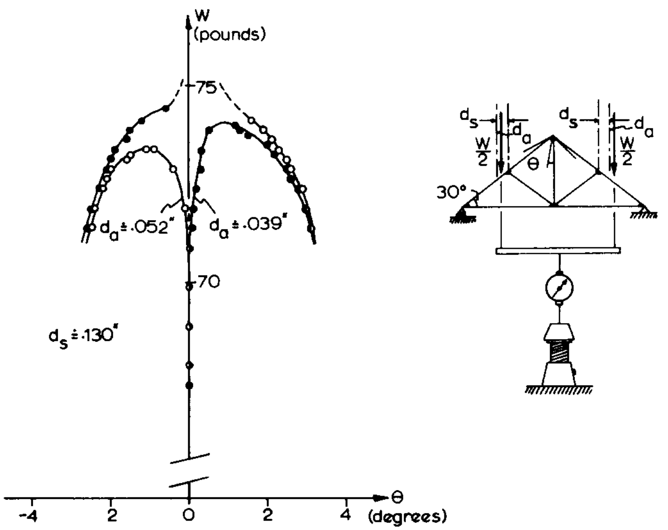

FIG. 5. Loading arrangement and experimental load deflection curves for the roof truss.

The first step in the experiment was to determine the load-deflection curves. The rotation, θ, of the top joint was used as a deflection parameter. The curves obtained are plotted in Fig. 5. Two sets of curves for slightly different anti-symmetrical eccentricities were obtained to bound the area in which the ideal post-buckling curve must lie. The curves clearly indicate that the symmetrically loaded roof truss constitutes an unstable-symmetric system. The symmetry result is to be expected if one examines the buckling mode of the system. From the photograph in Fig. 6 it is readily deduced that the configuration of negative buckling must be the mirror image of positive buckling. The potential energy for the ideal system must therefore be the same in both cases and hence a symmetrical post-buckling behaviour should result.

In the second stage of the experiment, the variation of the critical load with anti-symmetrical eccentricities was investigated. Plots of the maximum load W^*, against d_a are shown in Fig. 7 for three constant values of d_s. All three curves exhibit the characteristic

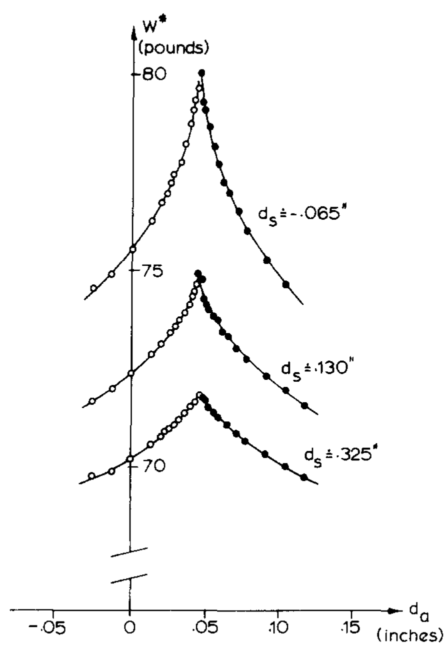

FIG. 7. Experimental W^*–d_a curves for constant values of d_s.

cusp of symmetric systems and all have an offset from the load axis to account for other (unknown) imperfections in the system. This offset is nearly the same for the three values of d_s, indicating that for this particular structure the quantity corresponding to C in equation (18b) is very small. The cusp tends to flatten out as the symmetric eccentricity d_s, changes in the positive direction.

The third step was to determine the variation in critical load with d_s at constant values of d_a. The experiment was carried out at four different values of d_a. The resulting curves are shown in Fig. 8. The curves tend to be more or less parallel with a finite slope, indicating that symmetrical eccentricities do have an appreciable influence if they are large enough.

From Fig. 7 and Fig. 8 it can be deduced that a three dimensional W^*–d_s–d_a plot would be a surface with a sharp ridge that slopes in the W^*–d_s plane. This partly confirms the conclusions reached in the preceding section. The curvature of the ridge in the W^*–d_a

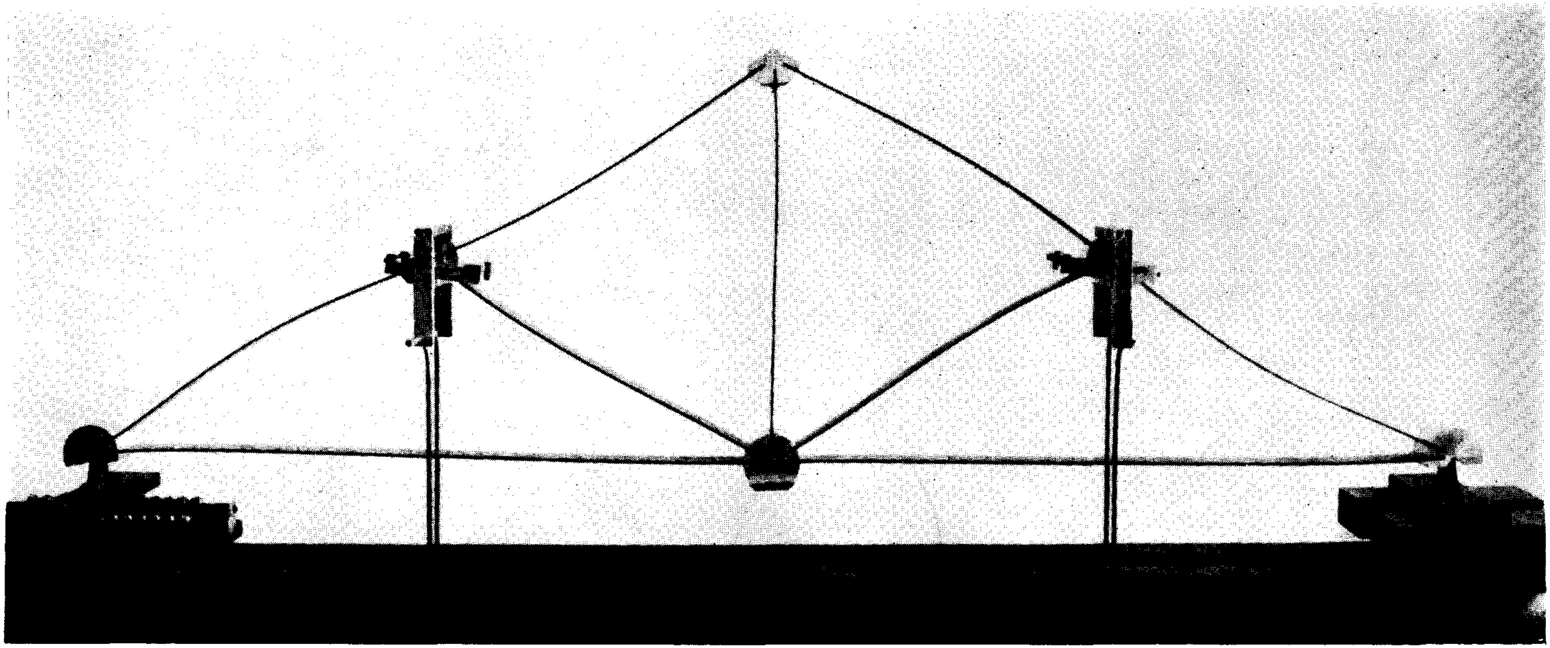

FIG. 6. The buckled structure.

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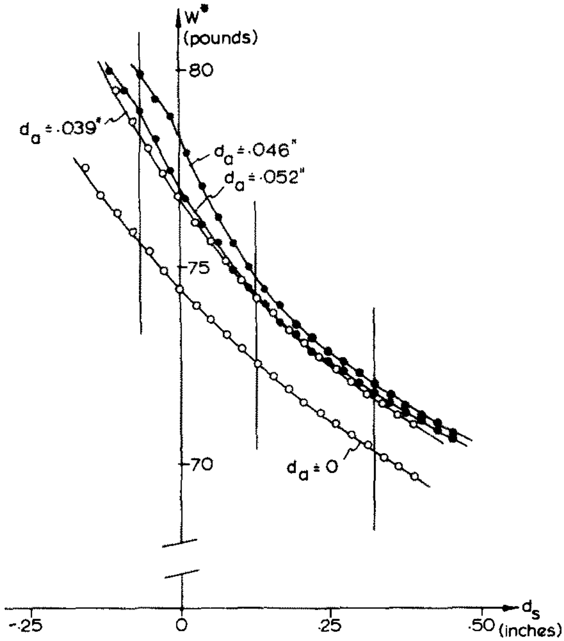

FIG. 8. Experimental W^*–d_s curves for constant values of d_a.

plane is very small (if it exists at all) and is hardly noticeable in the range of eccentricities allowed by the moveable knife-seat mechanism. It was therefore not found possible to obtain the cusps predicted by Fig. 3(a).

In this respect the small pre-buckling deformations of the truss due to axial shortening of the members may have affected the results to some extent. It is thought that this type of deformation introduces an additional "imperfection" of the second order type which may interact with d_s and thus distort the curves somewhat. To obtain a further insight into this problem a frame analysis of the Koiter type [4], which admits axial deformations, would be needed. This is, however, beyond the scope of the present paper.

6. CONCLUSIONS

(1) If both first and second order imperfections are present in a symmetrical structural system, there is a complex interaction between the two types of imperfection. Their combined effect on the buckling behaviour and critical load is generally dominated by the first order imperfection. The results obtained from an experiment on a roof truss give a clear picture of the relative effects of the two classes of imperfections on the critical load. The results show reasonably good qualitative agreement with theory for an unstable-symmetric system.

(2) The results obtained are of practical importance to structural engineers in two respects. First, and perhaps most important, great care must be taken in both the manufacturing of the system and the positioning of the loads to ensure that imperfections and eccentricities belonging to the first order class are reduced to a minimum. Secondly, there

may exist the possibility of increasing the critical load of unstable-symmetric systems by judiciously introducing certain imperfections or eccentricities of the second order type.

(3) It should be pointed out that it was necessary to perform the experiments at quite large effective imperfections due to the difficulty of obtaining consistent results at smaller values. Since the theory is valid only in the limit as the parameters approach zero, it is to be expected that the experimental curves should become somewhat distorted as the effective imperfections become larger. However on examining the experimental results closely, there seems to be very little distortion as far as the major imperfection is concerned, indicating that (at least for structures of the type examined here) the approximate theory is valid over quite a wide range of major imperfections.

(4) Finally, upon comparison of the experimental work on a roof truss reported here with that on a bridge truss reported earlier [5], a curious point emerges. Although in both cases the nominal system (i.e. frame and loading) is spatially symmetrical, the overall post-buckling behaviour is not of the same type. The bridge truss experiment yields an asymmetric load-deflection behaviour with its corresponding stable and unstable states, whereas the roof truss yields a completely unstable symmetric type behaviour. The explanation for this apparent inconsistency must be sought in the spatial mode of buckling. Whereas the spatial configurations of positive and negative buckling in the roof truss experiment possess the property of *mirror symmetry*, this is not the case in the bridge truss experiment. For this reason an asymmetric load-deflection behaviour may be expected in the latter case.

On the basis of these experiments, and others reported in Ref. [2], the following tentative rule for frame type structures may be formulated: If the system geometry possesses a line of symmetry, *and* if the spatial configuration of positive buckling is the mirror image of negative buckling, then the system produces a symmetric load-deflection behaviour in the post-buckling range.

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Абстракт—Исследуется устойчивость симметрических строительных систем с неточностями первого и второго рода. Анализ касается дискретной системы с "*n*" степенями свободы. Результирующие выражения, касающиеся уменьшения критической нагрузки относительно величин неточностей, указывают на существование комплексного взаимодействия между зтими величинами. Неточности второго рода имеют относительно меньшие влияние на критическую нагрузку. Обсуждаются зксперименты на жестко соединенных стропильных перекрытях. Зкспериментальные результаты показывают хоромую качественную сходимость с теорией.